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Some congruences modulo 2, 8 and 12 for Andrews' singular overpartitions

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Abstract. Recently, G. E. Andrews defined combinatorial objects which he called (k, i) -singular overpartitions, overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. Let the number of (k, i) -singular overpartitions of n be denoted by $\overline{C}_{k,i}(n)$. Andrews and Chen, Hirschhorn and Sellers noted numerous congruences modulo 2 for $\overline{C}_{3,1}(n)$. The object of this paper is to obtain new congruences modulo 2 for $\overline{C}_{20,5}(n)$ and modulo 8 and 12 for $\overline{C}_{3,1}(n)$.

Keywords: singular overpartition, congruence, generating function, sums of squares.

MSC 2000 classification: primary 05A17, secondary 11P83

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n . Let $p(n)$ be the number of partitions of n . For example $p(5) = 7$. The seven partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}} = \frac{1}{f_1},$$

where as customary, we define $f_k := (q^k; q^k)_{\infty} = \prod_{m=1}^{\infty} (1 - q^{mk})$.

If l is a positive integer, then a partition of n is said to be l -regular if no part

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is divisible by l . If $b_l(n)$ denotes the number of l -regular partitions of n then

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{(q^l; q^l)_{\infty}}{(q; q)_{\infty}} = \frac{f_l}{f_1}.$$

Several interesting arithmetic properties of l -regular partitions are found by many mathematicians, see [2, 6, 10, 11, 15, 19, 21]. In [9], Corteel and Lovejoy developed a new aspect of the theory of partitions - overpartitions. A hint of such a subject can also be seen in Hardy and Ramanujan [13, p.304]. An overpartition of n is a non-increasing sequence of positive integers whose sum is n in which the first occurrence of a part may be overlined. If $\bar{p}(n)$ denotes the number of overpartitions of n then

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} = \frac{f_2}{f_1^2}. \quad (1.1)$$

Lovejoy [17] investigated the function $\bar{A}_l(n)$ which counts the number of l -regular overpartitions of n . He also proved theorems for overpartitions analogous to Gordons celebrated generalization of the RogersRamanujan identities [12]. The generating function for $\bar{A}_l(n)$ is

$$\sum_{n=0}^{\infty} \bar{A}_l(n)q^n = \frac{(-q; q)_{\infty}(q^l; q^l)_{\infty}}{(q; q)_{\infty}(-q^l; q^l)_{\infty}} = \frac{f_2 f_l^2}{f_1^2 f_{2l}}.$$

Recently, G. E. Andrews [3] introduced (k, i) -singular overpartitions, overpartitions in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. Let $\bar{C}_{k,i}(n)$ denote the number of such partitions of n . For example, $\bar{C}_{3,1}(4) = 10$. The ten $(3, 1)$ -singular overpartitions of 4 are 4, $\bar{4}$, $2 + 2$, $\bar{2} + 2$, $2 + 1 + 1$, $\bar{2} + 1 + 1$, $2 + \bar{1} + 1$, $\bar{2} + \bar{1} + 1$, $1 + 1 + 1 + 1$ and $\bar{1} + 1 + 1 + 1$. The generating function for $\bar{C}_{k,i}(n)$, where $k \geq 3$ and $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ is

$$\sum_{n=0}^{\infty} \bar{C}_{k,i}(n)q^n = \frac{(q^k; q^k)_{\infty}(-q^i; q^k)_{\infty}(-q^{k-i}; q^k)_{\infty}}{(q; q)_{\infty}}. \quad (1.2)$$

In his paper [3], Andrews also proved that for $n \geq 0$,

$$\bar{C}_{3,1}(9n + 3) \equiv \bar{C}_{3,1}(9n + 6) \equiv 0 \pmod{3}. \quad (1.3)$$

It is important to note that $\bar{C}_{3,1}(n) = \bar{A}_3(n)$. Later, Chen, Hirschhorn and Sellers [8] found infinite families of congruences modulo 3 for $\bar{C}_{3,1}(n)$, $\bar{C}_{6,1}(n)$, $\bar{C}_{6,2}(n)$ and parity results for $\bar{C}_{4,1}(n)$. For example, they proved the following congruences,

Theorem 1.1. Let $p \equiv 3 \pmod{4}$ be prime. Then for all $k, m \geq 0$ with $p \nmid m$,

$$\overline{C}_{3,1}(p^{2k+1}m) \equiv 0 \pmod{3}. \quad (1.4)$$

In Theorem 1.1 if we set $p = 3, k = 0$ and $m \equiv 1, 2 \pmod{3}$, we can easily obtain (1.3). For recent works on singular overpartitions, see [1, 3, 7, 8, 16, 18, 20, 22]. The aim of this paper is to prove new congruences for $\overline{C}_{3,1}(n)$ and $\overline{C}_{20,5}(n)$. The following are our main results.

Theorem 1.2. For all $k, n \geq 0$,

$$\overline{C}_{3,1}(4^k(72n + 21)) \equiv 0 \pmod{12}, \quad (1.5)$$

$$\overline{C}_{3,1}(4^k(72n + 39)) \equiv 0 \pmod{12}, \quad (1.6)$$

$$\overline{C}_{3,1}(4^k(72n + 57)) \equiv 0 \pmod{12}. \quad (1.7)$$

Theorem 1.3. Let $p \geq 5$ be prime and $1 \leq s \leq p-1$ with $6s+1$ a quadratic nonresidue modulo p . Then, for all $m \geq 0$,

$$\overline{C}_{3,1}(18(pm + s) + 3) \equiv 0 \pmod{12}. \quad (1.8)$$

Theorem 1.4. For all $k, n \geq 0$,

$$\overline{C}_{3,1}(4^k(12n + 5)) \equiv 0 \pmod{8}, \quad (1.9)$$

$$\overline{C}_{3,1}(4^k(12n + 11)) \equiv 0 \pmod{8}. \quad (1.10)$$

Theorem 1.5. Let $p \geq 5$ be prime. Then for all $\alpha \geq 1$ and $n \geq 0$,

$$\overline{C}_{3,1}(48p^{2\alpha}n + (48j + 2p)p^{2\alpha-1}) \equiv 0 \pmod{8}, \quad j = 1, 2, \dots, p-1. \quad (1.11)$$

Theorem 1.6. For all $\alpha, n \geq 0$,

$$\overline{C}_{20,5}\left(2 \cdot 5^{2\alpha+1}n + \frac{31 \cdot 5^{2\alpha} - 7}{12}\right) \equiv 0 \pmod{2}, \quad (1.12)$$

$$\overline{C}_{20,5}\left(2 \cdot 5^{2\alpha+1}n + \frac{79 \cdot 5^{2\alpha} - 7}{12}\right) \equiv 0 \pmod{2}, \quad (1.13)$$

$$\overline{C}_{20,5}\left(2 \cdot 5^{2\alpha+2}n + \frac{83 \cdot 5^{2\alpha+1} - 7}{12}\right) \equiv 0 \pmod{2}, \quad (1.14)$$

$$\overline{C}_{20,5}\left(2 \cdot 5^{2\alpha+2}n + \frac{107 \cdot 5^{2\alpha+1} - 7}{12}\right) \equiv 0 \pmod{2}. \quad (1.15)$$

For an odd prime , the Legendre symbol is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1 & \text{if } a \text{ is quadratic non-residue modulo } p, \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

We also prove the following infinite family of congruences for $\overline{C}_{20,5}(n)$.

Theorem 1.7. Let $p \geq 5$ be prime, $\left(\frac{-10}{p}\right) = -1$. Then for all $\alpha, n \geq 0$,

$$\overline{C}_{20,5} \left(2p^{2\alpha+1}(pn+j) + 7 \times \frac{p^{2\alpha+2}-1}{12} \right) \equiv 0 \pmod{2}, \quad j = 1, 2, \dots, p-1. \quad (1.16)$$

In order to prove our main results, we collect a few definitions and Lemmas in section 2. In section 3-5 we prove our main results.

2 Preliminaries

We require the following definitions and lemmas to prove the main results in the next three sections.

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined as

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \quad (2.1)$$

Using Jacobi's triple product identity [5, Entry 19, p. 35], (2.1) becomes

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}. \quad (2.2)$$

The most important special cases of $f(a, b)$ are

$$\varphi(q) := f(q, q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty} = \frac{f_2^5}{f_1^2 f_4^2}, \quad (2.3)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \frac{f_2^2}{f_1} \quad (2.4)$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty} = f_1. \quad (2.5)$$

By the binomial theorem, we see that for any positive integer k ,

$$f_1^{2^k} \equiv f_2^{2^{k-1}} \pmod{2^k}. \quad (2.6)$$

Lemma 2.1. (Hirschhorn, Garvan and Borwein [14]) The following 2-dissection holds

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}. \quad (2.7)$$

Lemma 2.2. (Hirschhorn and Sellers [15, Theorem 2.1, 2.3, 2.4]) We have,

$$\sum_{n=0}^{\infty} b_5(n) q^n = \frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}, \quad (2.8)$$

$$\sum_{n=0}^{\infty} b_5(2n+1) q^n \equiv \frac{f_5 f_{20}}{f_1 f_{10}} \pmod{2}, \quad (2.9)$$

$$b_5(20n+5) \equiv 0 \pmod{2}, \quad (2.10)$$

$$b_5(20n+13) \equiv 0 \pmod{2}. \quad (2.11)$$

Lemma 2.3. (Cui and Gu [10, Theorem 2.2]) If $p \geq 5$ is a prime and

$$\frac{\pm p - 1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}, \end{cases}$$

then

$$\begin{aligned} (q; q)_{\infty} &= \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f \left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}} \right) \\ &\quad + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2}; q^{p^2})_{\infty}. \end{aligned} \quad (2.12)$$

Furthermore, if $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$, $k \neq \frac{\pm p-1}{6}$ then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}$.

3 Congruences modulo 12 for $\overline{C}_{3,1}(n)$

In this section we prove some infinite families of congruences modulo 12 for $\overline{C}_{3,1}(n)$.

From [19, Theorem 2.7, Eq. 2.22], we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(9n+3) q^n = 6 \frac{f_2^8 f_3^{15}}{f_1^{17} f_6^6} + 96q \frac{f_2^5 f_3^6 f_6^3}{f_1^{14}}. \quad (3.1)$$

Using (2.6), we have

$$\frac{f_2^8 f_3^{15}}{f_1^{17} f_6^6} \equiv \frac{f_3^3}{f_1} \pmod{2}. \quad (3.2)$$

Using (3.2) in (3.1), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(9n+3)q^n \equiv 6 \frac{f_3^3}{f_1} \pmod{12}. \quad (3.3)$$

Substituting the identity (2.7) in (3.3) and then simplifying, we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(9n+3)q^n \equiv 6f_8 + 6q \frac{f_{12}^3}{f_4} \pmod{12}. \quad (3.4)$$

Equating the coefficients of q^{4n+1} from both sides of (3.4), dividing both sides by q and then replacing q^4 by q , we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(36n+12)q^n \equiv 6 \frac{f_3^3}{f_1} \equiv \sum_{n=0}^{\infty} \overline{C}_{3,1}(9n+3)q^n \pmod{12}, \quad (3.5)$$

which yields,

$$\overline{C}_{3,1}(36n+12) \equiv \overline{C}_{3,1}(9n+3) \pmod{12}. \quad (3.6)$$

By (3.6) and mathematical induction, we find that for $n, k \geq 0$,

$$\overline{C}_{3,1}(4^k(9n+3)) \equiv \overline{C}_{3,1}(9n+3) \pmod{12}. \quad (3.7)$$

Equating the coefficients of q^{2n} from both sides of (3.4), and then replacing q^2 by q , we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(18n+3)q^n \equiv 6f_4 \pmod{12}. \quad (3.8)$$

Equating the coefficients of q^{4n+1} , q^{4n+2} , q^{4n+3} from the both sides of (3.8), we obtain

$$\overline{C}_{3,1}(72n+21) \equiv 0 \pmod{12}, \quad (3.9)$$

$$\overline{C}_{3,1}(72n+39) \equiv 0 \pmod{12}, \quad (3.10)$$

$$\overline{C}_{3,1}(72n+57) \equiv 0 \pmod{12}. \quad (3.11)$$

Proof of Theorem 1.2. Replacing n by $8n+2$ in (3.7) and using (3.9), we obtain (1.5). Replacing n by $8n+4$ in (3.7) and using (3.10), we have (1.6). Replacing n by $8n+6$ in (3.7) and then employing (3.11), we obtain (1.7). \square

Theorem 3.1. For all $n \geq 0$,

$$\overline{C}_{3,1}(18n+3) \equiv \begin{cases} 6 & (\text{mod } 12) \text{ if } n = 2k(3k-1), \\ 0 & (\text{mod } 12) \text{ otherwise.} \end{cases} \quad (3.12)$$

Proof. Using Euler's Pentagonal Number Theorem [4, p. 12] in (3.8), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(18n+3)q^n \equiv 6 \sum_{k=-\infty}^{\infty} q^{2k(3k-1)} \pmod{12}. \quad (3.13)$$

□*QED*

Proof of Theorem 1.3. Replacing q by q^6 in both sides of (3.13) and then multiplying both sides by q , we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(18n+3)q^{6n+1} \equiv 6 \sum_{k=-\infty}^{\infty} q^{12k(3k-1)+1} \pmod{12}, \quad (3.14)$$

which yields

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(18n+3)q^{6n+1} \equiv 6 \sum_{k=-\infty}^{\infty} q^{(6k-1)^2} \pmod{12}. \quad (3.15)$$

Let $n = pm + s$, then $6n+1 = 6pm+6s+1 \equiv 6s+1 \pmod{p}$ is not a quadratic residue modulo p . Thus, $6n+1$ is not a square and $\overline{C}_{3,1}(18n+3) \equiv 0 \pmod{12}$.

□*QED*

4 Congruences modulo 8 for $\overline{C}_{3,1}(n)$

In this section, we prove some arithmetic properties modulo 8 satisfied by $\overline{C}_{3,1}(n)$.

From [19, Theorem 2.6, Eq. 2.16], we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(3n+2)q^n = 4 \frac{f_2 f_6^3}{f_1^4}. \quad (4.1)$$

Using (2.6), it follows that

$$\frac{f_2 f_6^3}{f_1^4} \equiv \frac{f_3^6}{f_1^2} \pmod{2}. \quad (4.2)$$

Using (4.2) in (4.1), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(3n+2)q^n \equiv 4 \frac{f_3^6}{f_1^2} \pmod{8}. \quad (4.3)$$

Substituting the identity (2.7) in (4.3) and then using (2.6), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{3,1}(3n+2)q^n &\equiv 4 \left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4} \right)^2 \\ &\equiv 4 \left(\frac{f_4^6 f_6^4}{f_2^4 f_{12}^2} + q^2 \frac{f_{12}^6}{f_4^2} \right) \\ &\equiv 4 \left(f_4^4 + q^2 \frac{f_{12}^6}{f_4^2} \right) \pmod{8}. \end{aligned} \quad (4.4)$$

Extracting the terms containing q^{4n+2} from both sides of (4.4), dividing both sides by q^2 and then replacing q^4 by q , we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(12n+8)q^n \equiv 4 \frac{f_3^6}{f_1^2} \equiv \sum_{n=0}^{\infty} \overline{C}_{3,1}(3n+2)q^n \pmod{8}, \quad (4.5)$$

which yields,

$$\overline{C}_{3,1}(12n+8) \equiv \overline{C}_{3,1}(3n+2) \pmod{8}. \quad (4.6)$$

By (4.6) and mathematical induction we have, for $n, k \geq 0$,

$$\overline{C}_{3,1}(4^k(3n+2)) \equiv \overline{C}_{3,1}(3n+2) \pmod{8}. \quad (4.7)$$

Extracting the terms containing q^{4n} from both sides of (4.4) and then replacing q^4 by q we have,

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(12n+2)q^n \equiv 4f_1^4 \equiv 4f_4 \pmod{8}. \quad (4.8)$$

Equating the coefficients of q^{4n+1} and q^{4n+3} from both sides of (4.4), we have

$$\overline{C}_{3,1}(12n+5) \equiv 0 \pmod{8}, \quad (4.9)$$

$$\overline{C}_{3,1}(12n+11) \equiv 0 \pmod{8}. \quad (4.10)$$

Proof of Theorem 1.4. Replacing n by $4n+1$ in (4.7) and using (4.9), we obtain (1.9). Again by replacing n by $4n+3$ in (4.7) and using (4.10), we have (1.10).

QED

Theorem 4.1. Let $p \geq 5$ be prime. Then for all $\alpha, n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{C}_{3,1} (48p^{2\alpha}n + 2p^{2\alpha}) q^n \equiv 4(q; q)_{\infty} \pmod{8}. \quad (4.11)$$

Proof. Extracting the terms involving q^{4n} , from the both sides of (4.8) and then replacing q^4 by q , we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1} (48n + 2) q^n \equiv 4f_1 \pmod{8}, \quad (4.12)$$

which is the case $\alpha = 0$ of (4.11). Suppose that (4.11) is true for some $\alpha \geq 0$. Substituting (2.12) into (4.11), extracting the terms containing $q^{pn + \frac{p^2-1}{24}}$ from both sides of the identity, dividing both sides by $q^{\frac{p^2-1}{24}}$ and then replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1} \left(48p^{2\alpha} \left(pn + \frac{p^2-1}{24} \right) + 2p^{2\alpha} \right) q^n \equiv 4(q^p; q^p)_{\infty} \pmod{8}. \quad (4.13)$$

Extracting the terms containing of q^{pn} from both sides of (4.13) and then replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1} \left(48p^{2(\alpha+1)}n + 2p^{2(\alpha+1)} \right) q^n \equiv 4(q; q)_{\infty} \pmod{8}, \quad (4.14)$$

which is (4.11) with $\alpha + 1$ for α . This completes the proof of (4.11) by induction. \square

Proof of Theorem 1.5. Comparing the coefficients of q^{pn+j} , for $1 \leq j \leq p-1$, from both sides of (4.13), we arrive at (1.11). \square

5 Congruences modulo 2 for $\overline{C}_{20,5}(n)$

In this section, we prove a number of arithmetic properties modulo 2 satisfied by $\overline{C}_{20,5}(n)$.

Theorem 5.1. For all $\alpha, n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2 \cdot 5^{2\alpha}n + 7 \times \frac{5^{2\alpha}-1}{12} \right) q^n \equiv (q^2; q^2)_{\infty} (q^5; q^5)_{\infty} \pmod{2}. \quad (5.1)$$

Proof. From (1.2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{20,5}(n)q^n &= \frac{(q^{20}; q^{20})_{\infty} (-q^5; q^{20})_{\infty} (-q^{15}; q^{20})_{\infty}}{(q; q)_{\infty}} \\ &\equiv \frac{f_{10}f_5}{f_1} \pmod{2}. \end{aligned} \quad (5.2)$$

Using (2.6) in (2.8), we obtain

$$\begin{aligned} \frac{f_5}{f_1} &= \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \\ &\equiv f_4 + q \frac{f_{10} f_{20}}{f_2} \pmod{2}. \end{aligned} \quad (5.3)$$

Using (5.3) in (5.2), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}(n)q^n \equiv f_4 f_{10} + q \frac{f_{10}^2 f_{20}}{f_2} \pmod{2}. \quad (5.4)$$

Extracting the terms containing q^{2n} from both sides of (5.4) and then replacing q^2 by q , we have

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}(2n)q^n \equiv f_2 f_5 \pmod{2}, \quad (5.5)$$

which is the case $\alpha = 0$ of (5.1). Now suppose (5.1) holds for some $\alpha \geq 0$. Recall Ramanujan's beautiful identity [4, p. 161]:

$$\frac{f_1}{f_{25}} = R(q^5)^{-1} - q - q^2 R(q^5), \quad (5.6)$$

where

$$R(q) = \frac{f(-q, -q^4)}{f(-q^2, -q^3)}.$$

Replacing q by q^2 in (5.6), we get

$$\frac{f_2}{f_{50}} = R(q^{10})^{-1} - q^2 - q^4 R(q^{10}). \quad (5.7)$$

Using (5.7) in (5.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2 \cdot 5^{2\alpha} n + 7 \times \frac{5^{2\alpha} - 1}{12} \right) q^n &\equiv (q^2; q^2)_{\infty} (q^5; q^5)_{\infty} \pmod{2} \\ &= f_5 f_{50} \left(R(q^{10})^{-1} - q^2 - q^4 R(q^{10}) \right). \end{aligned} \quad (5.8)$$

Extracting the terms containing q^{5n+2} from both sides of (5.8), then dividing both sides by q^2 and finally replacing q^5 by q , we have

$$\sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2 \cdot 5^{2\alpha} (5n+2) + 7 \times \frac{5^{2\alpha} - 1}{12} \right) q^n \equiv f_1 f_{10} \pmod{2}. \quad (5.9)$$

Using (5.6) in (5.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2 \cdot 5^{2\alpha+1} n + \frac{11 \cdot 5^{2\alpha+1} - 7}{12} \right) q^n &\equiv f_1 f_{10} \pmod{2} \\ &= f_{10} f_{25} \left(R(q^5)^{-1} - q - q^2 R(q^5) \right). \end{aligned} \quad (5.10)$$

Extracting the terms containing q^{5n+1} from both sides (5.10), then dividing both sides by q and finally replacing q^5 by q , we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2 \cdot 5^{2\alpha+1} (5n+1) + \frac{11 \cdot 5^{2\alpha+1} - 7}{12} \right) q^n \\ = \sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2 \cdot 5^{2(\alpha+1)} n + 7 \times \frac{5^{2(\alpha+1)} - 1}{12} \right) q^n \equiv f_2 f_5 \pmod{2}, \end{aligned} \quad (5.11)$$

which is (5.1) with $\alpha + 1$ for α . This completes the proof of (5.1) by induction. \square

Proof of Theorem 1.6. Comparing the coefficients of q^{5n+1} and q^{5n+3} from both sides of (5.8), we obtain the first two congruences of Theorem 1.6. Comparing the coefficients of q^{5n+3} and q^{5n+4} from both sides of (5.10), we obtain the remaining two congruences of Theorem 1.6. \square

Theorem 5.2. Let $p \geq 5$ be prime, $\left(\frac{-10}{p} \right) = -1$. Then for all $\alpha, n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2p^{2\alpha} n + 7 \times \frac{p^{2\alpha} - 1}{12} \right) q^n \equiv (q^2; q^2)_{\infty} (q^5; q^5)_{\infty} \pmod{2}. \quad (5.12)$$

Proof. Now (5.5) is the $\alpha = 0$ case of (5.12). Suppose (5.12) is true for some $\alpha \geq 0$. Using (2.12) on the right hand of (5.12), we obtain

$$\begin{aligned}
& \sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2p^{2\alpha}n + 7 \times \frac{p^{2\alpha} - 1}{12} \right) q^n \\
& \equiv \left[\sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \pm \frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{2 \cdot \frac{3k^2+k}{2}} f \left(-q^{2 \cdot \frac{3p^2+(6k+1)p}{2}}, -q^{2 \cdot \frac{3p^2-(6k+1)p}{2}} \right) \right. \\
& \quad \left. + (-1)^{\frac{\pm p-1}{6}} q^{2 \cdot \frac{p^2-1}{24}} (q^{2p^2}; q^{2p^2})_{\infty} \right] \\
& \quad \times \left[\sum_{\substack{m=-\frac{p-1}{2} \\ m \neq \pm \frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{5 \cdot \frac{3m^2+m}{2}} f \left(-q^{5 \cdot \frac{3p^2+(6m+1)p}{2}}, -q^{5 \cdot \frac{3p^2-(6m+1)p}{2}} \right) \right. \\
& \quad \left. + (-1)^{\frac{\pm p-1}{6}} q^{5 \cdot \frac{p^2-1}{24}} (q^{5p^2}; q^{5p^2})_{\infty} \right] \pmod{2}. \tag{5.13}
\end{aligned}$$

For a prime p with $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$, let us consider

$$2 \cdot \frac{3k^2+k}{2} + 5 \cdot \frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod{p},$$

which equivalent to

$$(12k+2)^2 + 10(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-10}{p}\right) = -1$, the only solution of the above condition is $k, m = \frac{\pm p-1}{6}$.

Therefore extracting the terms containing $q^{pn+7 \cdot \frac{p^2-1}{24}}$ from both sides of (5.13),

then dividing both sides by $q^{7 \cdot \frac{p^2-1}{24}}$ and replacing q^{pn} by q we obtain,

$$\begin{aligned} & \sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2p^{2\alpha} \left(pn + 7 \cdot \frac{p^2-1}{24} \right) + 7 \times \frac{p^{2\alpha}-1}{12} \right) q^n \\ &= \sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2p^{2\alpha+1}n + 7 \times \frac{p^{2\alpha+2}-1}{12} \right) q^n \equiv (q^{2p}; q^{2p})_{\infty} (q^{5p}; q^{5p})_{\infty} \pmod{2}. \end{aligned} \quad (5.14)$$

Extracting the terms containing q^{pn} from both sides of (5.14) and then replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{20,5} \left(2p^{2(\alpha+1)}n + 7 \times \frac{p^{2(\alpha+1)}-1}{12} \right) q^n \equiv (q^2; q^2)_{\infty} (q^5; q^5)_{\infty} \pmod{2}, \quad (5.15)$$

which is (5.12) with $\alpha+1$ for α . This completes the proof of (5.12) by induction. \square

Proof of Theorem 1.7. Comparing the coefficients of q^{pn+j} , for $1 \leq j \leq p-1$ from both sides of (5.14), we arrive at (1.16). \square

We close this section by briefly noting the following corollary.

Corollary 1. For all $n \geq 0$,

$$\overline{C}_{20,5}(n) \equiv b_5(2n+1) \pmod{2}, \quad (5.16)$$

$$\overline{C}_{20,5}(10n+2) \equiv 0 \pmod{2}, \quad (5.17)$$

$$\overline{C}_{20,5}(10n+6) \equiv 0 \pmod{2}. \quad (5.18)$$

Proof. From (2.9) we have,

$$\sum_{n=0}^{\infty} b_5(2n+1)q^n \equiv \frac{f_5 f_{20}}{f_1 f_{10}} \equiv \frac{f_5 f_{10}}{f_1} \pmod{2}, \quad (5.19)$$

(5.16) follows from (5.2) and (5.19). Replacing n by $10n+2$ in (5.16), using (2.10), we have (5.17). Again replacing n by $10n+6$ in (5.16) and using (2.11), we obtain (5.18). Observe that (5.17) is the $\alpha=0$ case of (1.12) and (5.18) is the $\alpha=0$ case of (1.13). \square

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